W2L3 - NON-LINEAR MODELS

Population Dynamics

Recall: The Malthusian Model: We assume population growth is proportional to the population:

$$\frac{dP}{dt} = kP \Rightarrow \frac{\frac{dP}{dt}}{P} = k$$

↑ constant

What if the ratio between the rate and the population is NOT constant, but depends on other time-independent mechanisms?

$$\frac{\left(\frac{dP}{dt}\right)}{P} = f(P)$$

↑ only depends on P

Logistic Equation ← (Assuming f is linear)

Assume that a population can have no more than N individuals, i.e. carrying capacity

with respect to the model

$\Rightarrow \frac{dP}{dt}$ is jointly proportional to P and N-P

$$\Rightarrow \frac{dP}{dt} \propto P(N-P) \Rightarrow \boxed{\frac{dP}{dt} = kP(N-P)}$$

Logistic Eq. (1)

OR:

$$\frac{\frac{dP}{dt}}{P} = a - bP \Rightarrow \boxed{\frac{dP}{dt} = P(a-bP)}$$

Logistic Eq. (2)

$a = kN$

$b = k$

General Logistic Equation: $\frac{dP}{dt} = P(a-bP)$

Solution to the General Logistic Equation:

$$\frac{dP}{dt} = P(a-bP)$$

← autonomous (time independent)
↳ separable

Note: the equation is separable. So solving should be easy.

$$\int \frac{dP}{P(a-bP)} = \int dt$$

We need to use Partial Fractions.

EX: Integrate $\int \frac{6}{x^2+2x-8}\,dx = \int \frac{6}{(x+4)(x-2)}\,dx$

$$\left(\frac{6}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2}\right)(x+4)(x-2)$$

$$6 = A(x-2) + B(x+4)$$

$x=2 \rightarrow 6 = B(6) \Leftrightarrow B = 1$

$x=-4 \rightarrow 6 = A(-6) \Leftrightarrow A = -1$

$$\hookrightarrow \int \frac{-1}{x+4} + \frac{1}{x-2}\,dx$$

$$= -\ln|x+4| + \ln|x-2| + C$$

$$= \boxed{\ln\left|\frac{x-2}{x+4}\right| + C}$$

$$\int \frac{dP}{P(a-bP)} = \int dt$$

$$\frac{1}{P(a-bP)} = \frac{A}{P} + \frac{B}{a-bP}$$

$$1 = A(a-bP) + BP$$

Let $P=0 \Rightarrow 1 = A \cdot a \Rightarrow A = \frac{1}{a}$

Let $P = \frac{a}{b} \Rightarrow 1 = B(\frac{a}{b}) \Rightarrow B = \frac{b}{a}$

$$\int \frac{1}{P(a-bP)}\,dP = \int \frac{1/a}{P} + \frac{b}{a(a-bP)}\,dP = \frac{1}{a}\int \frac{1}{P}\,dP + \frac{b}{a}\underbrace{\int \frac{1}{a-bP}\,dP}_{\substack{u = a-bP \\ du = -b\,dP}}$$

$$= \frac{1}{a}\ln P - \frac{1}{a}\ln(a-bP) = \int dt = t + C$$

Now we have the following

$$\frac{1}{2}\ln P - \frac{1}{a}\ln(a-bP) = t + C$$

Solve for P:

$$\ln P - \ln(a-bP) = at + C_1$$

$$\ln \frac{P}{a-bP} = at + C_1 \qquad C_2 = e^{C_1}$$

Multi $a-Pb$ $\left(\frac{P}{a-bP} = C_2 e^{at}\right) \Rightarrow P = Ce^{at}(a-bP) = Ce^{at} - CbPe^{at} + CbPe^{at}$

$$\Rightarrow P(1 + Cbe^{at}) = Cae^{at}$$

$$P = \frac{Cae^{at}}{1 + cbe^{at}} \cdot \frac{e^{-at}}{e^{-at}} = \frac{Ca}{e^{-at} + Cb}$$

We can finally write:

$$P(t) = \frac{ac}{bc + e^{-at}}$$

Consider the initial conditions, $P(0) = P_0$. Substitute this into equation (2):

$$\Rightarrow P_0 = \frac{aC}{bC + 1} \Rightarrow P_0 bC + P_0 = ac$$

$$P_0 = ac - P_0 bc$$

$$P_0 = c(a - bP_0)$$

$$\Rightarrow c = \frac{P_0}{a - bP_0}$$

$$P(t) = \frac{a\left(\frac{P_0}{a-bP_0}\right)}{b\left(\frac{P_0}{a-bP_0}\right) + e^{-at}}$$

The logistic equation:

$$P(t) = \frac{aP_0}{bP_0 + (a-bP_0)e^{-at}}$$

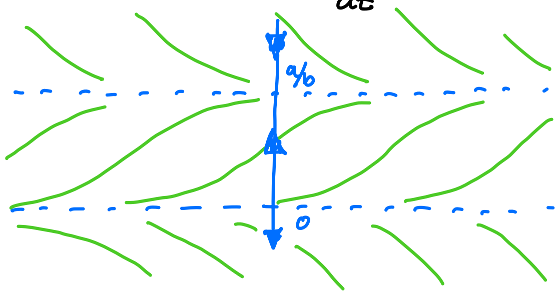
Note: $\lim_{t\to\infty} \frac{aP_0}{bP_0} = \frac{a}{b}$

Recall: $\frac{dP}{dt} = kP(N-P)$

which implies that $a = kN$ and $b = k$

$$\Rightarrow \lim_{t\to\infty} P(t) = \frac{a}{b} = \frac{kN}{k} = N$$

Recall Phase Portrait of $\frac{dP}{dt} = P(a-bP)$



EX

Student with flu returned to a isolated campus of 1000 students. If infection rate proportional to both # of infected (x) but # of not infected, determine the number of infected students after 6 days if x(4) = 50.

$$\begin{cases} \frac{dx}{dt} = kx(1000-x) \\ \frac{dx}{dt} = x(1000k - kx) \\ x(0) = 1 \end{cases}$$

$\frac{dP}{dt} = P(a-bP)$

$P(t) = \frac{aP_0}{bP_0 + (a-bP_0)e^{-at}}$

$a = 1000k \qquad b = k \qquad x_0 = 1 \leftarrow P_0$

$$x(t) = \frac{1000k(1)}{k(1) + (1000k - k(1))e^{-1000kt}}$$

$$= \frac{1000\cancel{k}}{\cancel{k} + 999\cancel{k}e^{-1000kt}} = \frac{1000}{1+999e^{-1000kt}}$$

$x(4) = 50$

$$50 = \frac{1000}{1+999e^{-4000k}}$$

$\vdots$

$-1000k = -0.9906$

$$x(t) = \frac{1000}{1+999e^{-0.9906t}}$$

$$x(6) = \frac{1000}{1+999e^{-0.9906(6)}} \approx \underline{276 \text{ students}}$$